Journal of Algebra 636 (2023) 510–532

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Globalization of partial group actions on semiprime Lie algebras and unital Jordan algebras

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A R T I C L E   I N F O

Article history:
Received 15 May 2023
Available online 15 September 2023
Communicated by Alberto Elduque

MSC:
17A36
17B99
17C99
17A99

Keywords:
Lie algebra
Jordan algebra
Partial group action
Globalization

A B S T R A C T

We give necessary and sufficient conditions for the existence of a semiprime globalization for a partial group action on a semiprime Lie algebra L, and with an additional reasonable condition, we show that this semiprime globalization is unique up to isomorphism. Moreover, under the same condition we prove that any globalizable partial group action on L induces a globalizable partial group action on its maximal quotient algebra. For Jordan algebras, we show that a globalizable partial group action on a unital Jordan algebra J induces a globalizable partial group action on the unital special universal envelope for J.

1. Introduction

Partial actions, in the sense we are dealing with (see [26]), can be found in various areas of mathematics (see [36]), formerly being used independently and often without any specific name. Their origin goes back to the 1800s due to the fact that a flow of

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https://doi.org/10.1016/j.jalgebra.2023.09.009
0021-8693/

a differentiable vector field on a manifold is a genuine partial action of the additive group of the real numbers [1]. Flows in differential geometry are important examples of local transformation groups, and the notion of a *maximum local transformation group* given in R. S. Palais's memoir [45, p. 65] is indeed the concept of a partial group action formulated in the differentiable context.

The term *partial action* was used in close but different senses. In the meaning we employ it, the concept was defined in the theory of C^*-algebras, with the desire to describe relevant classes of C^*-algebras as more general crossed products (see [24], [40], [25], [26]). Since, a number of important classes of C^*-algebras were shown to have partial crossed product structures, allowing one to obtain significant results on their representations, ideal structure and K-theory. Independently, in [33] a very close concept was defined under the same term "partial action", when dealing with graded modules for quotients of path algebras.

Influenced by C^*-algebraic advances purely algebraic developments occurred in various directions. One of them is the study of the globalization problem for partial actions, which can be illustrated by the fact of the Möbius group acting globally on the Riemann sphere and only partially on the complex plain [36]. In fact, the partial action on the complex plain can be seen as the restriction of the global action on the sphere, and the globalization problem asks whether or not it is possible to identify a given partial action with a restriction of a global one. Involving infinitesimal group actions this problem was considered in the above mentioned memoir by R. S. Palais for partial actions of Lie groups on (non-necessarily Hausdorff) smooth manifolds (see [45, Theorem X, p. 72]). Later, it was independently studied by F. Abadie in [1] for partial group actions on topological spaces and on C^*-algebras, by B. Steinberg in [50] for partial group actions on cell-complexes, and by J. Kellendonk and M. V. Lawson in [36] for partial group actions on topological spaces and other structures. In a purely algebraic context of partial group actions on algebras, the globalization problem was addressed in [22], where, in particular, a necessary and sufficient condition for the existence and uniqueness (up to isomorphism) of a globalization for a partial action of a group on a unital associative algebra was provided. The globalization problem also was studied for partial group actions on non-unital algebras in [10], [15] and [21]. Further developments on globalizations of partial group actions can be found in [2,23,28,29,37,48]. In addition, this problem was also investigated for partial actions of semigroups [32,34,38,43], groupoids [6,7,31,39], categories [16,44] and in the context of partial (weak) Hopf (co)actions [3–5,13,14].

The defining feature of the "partiality" we are dealing with, is that the composition of the actions of two elements is only a restriction of the action of their product (see Remark 2.2 or, more generally, [27, Definition 2.1]). If the composition equals to the action of the product, we say that the action is global (or usual). Note that usual actions of groupoids (see [31]) may be still partial in a sense: the action of an element might be only partially defined on some subset of the set under the action. Groupoid actions were called pre-actions in [41,42] and studying them on topological and metric spaces, considering this very natural kind of partiality, a universal globalization was constructed in [41], which possesses several interesting properties (see also [42]). A remarkable point is that the two types of partiality for groupoid actions collapse into a single one for groups and the same may be said about inverse semigroups versus groups. Due to the different sense of partiality the concept of a globalization in [41,43] differs from the one we mentioned above, nevertheless the construction used in [41,42] is essentially the same as that employed for partial group actions on topological spaces, cell-complexes etc., and which has been rediscovered several times in various areas (see [36]). Observe also that for globalization of partial actions on algebras a different construction is needed [22]. Yet another procedure was employed by F. Abadie in [1] for globalization of partial group actions on C^*-algebras up to Morita equivalence (see [2] for an analogue in the case of algebras over commutative rings). For more information around partial actions and their applications we refer the reader to R. Exel's book [27] and the surveys [8], [19] and [20].

More recently, the globalization problem for partial group actions on semisimple and reductive Lie algebras was studied in [46] and for some classes of Lie algebras, Jordan algebras and Malcev algebras in [17]. In the present paper we investigate the globalization problem for partial group actions on semiprime Lie algebras and on unital special universal envelopes for unital Jordan algebras. With respect to semiprime Lie algebras we also consider the question of the extension of a partial group action on a Lie algebra L to a partial action on the maximal algebra of quotients of L. The latter was introduced by M. Siles Molina in [47], followed by many interesting developments and applications (see, in particular, [11,12,49]).

The structure of this article is as follows. Section 2 summarizes the definitions and some known results about partial group actions on non-associative algebras that will be used in the rest of the paper. Then in Sections 3 and 4 we give a brief study of the structure of a semiprime Lie algebra and its maximal algebra of quotients. The presented facts are inspired by their known analogues for semiprime rings. The main result in Section 5 is the following theorem concerning the globalization problem of partial group actions, which can be considered as a Lie analogue of a similar result known in the case of semiprime rings (see [15]), which in its turn was influenced by [21, Theorem 3.1]. We recall that an ideal I of a semiprime Lie algebra L is called closed if $C_L(C_L(I)) = I$ (see Section 3).

Theorem 1.1. *Let L be a semiprime (respectively, non-degenerate) Lie algebra. Suppose that $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial action of G on L, such that every D_g is a closed ideal. Then, α possesses a globalization (η, H) if and only if for each $a \in L$ and $g \in G$ there is $\gamma_g(a) \in \operatorname{Der}(L)$ satisfying:*

(1) $\gamma_g(a)(L) \subseteq D_g$, and
(2) $\gamma_g(a) = \alpha_g \circ \operatorname{ad} a \circ \alpha_{g^{-1}}$ on D_g.

Moreover, in this situation, it is possible to choose a semiprime (respectively, non-degenerate) globalization (η, H) of α.

As an important consequence of Theorem 1.1 we have Corollary 5.4, which gives a simple condition for the existence and uniqueness of a globalization for a partial group action on a semiprime Lie algebra. In the same section we prove that a partial group action $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on L, where every D_g is a direct factor of L, induces, in a natural way, a partial group action $\alpha^* = (\{Q_g\}_{g\in G}, \{\alpha^*_g\}_{g\in G})$ on the maximal algebra of quotients of L. Moreover, α^* admits a semiprime globalization that is unique up to isomorphism. This result is a consequence of Proposition 5.1 and Corollary 5.6.

Section 6 is devoted to the partial group actions on the unital special universal envelope for a Jordan algebra. Given a unital Jordan algebra J, the unital special envelope for J is a unital associative algebra A together with a unital Jordan homomorphism $J \to A^+$ which is universal in a natural sense (see Section 6). The main result of the section is the following fact, which is a consequence of Proposition 6.2 and Theorem 6.3.

Theorem 1.2. *Let $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on a unital Jordan algebra J, where each D_g is a unital ideal of J, and let (A, σ) be the unital special universal envelope of J. Then the following holds.*

(1) If A_g denotes the subalgebra of A generated by $\sigma(D_g)$ and $\widetilde{\alpha}_g : A_{g^{-1}} \to A_g$ denotes the isomorphism induced by α_g, then $\widetilde{\alpha} = (\{A_g\}_{g\in G}, \{\widetilde{\alpha}_g\}_{g\in G})$ is a partial action of G on A.

(2) Suppose that α is of finite type (see Section 2). If (η, K) is a globalization for α and (B, σ) is the unital special universal envelope for K, then $(\widetilde{\eta}, B)$ is a globalization for $\widetilde{\alpha}$.

2. Partial actions on non-associative algebras

Throughout this paper, G denotes an arbitrary group, whose neutral element will be denoted by e. Furthermore, all considered algebras are defined over a field of arbitrary characteristic, unless otherwise stated.

In this section $\mathcal{C}$ denotes any of the following categories: the category of Lie algebras, the category of the Jordan algebras, or the category of associative algebras; and we will write, slightly abusing the notation, $A \in \mathcal{C}$ to mean that A is an object of the category $\mathcal{C}$. In the next paragraphs, we recall the concept of a partial action of a group on an algebra $A \in \mathcal{C}$ and that of its globalization.

Definition 2.1. Let $A \in \mathcal{C}$. A *partial action* of G on A is a pair of collections

$$\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}),$$

where, for each $g \in G$, D_g is an ideal of A (obviously $D_g \in \mathcal{C}$) and $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism, satisfying for all $g, h \in G$:

(1) $D_e = A$ and α_e is the identity map of A;

(2) $\alpha_g(D_{g^{-1}} \cap D_h) \subseteq D_{gh}$;
(3) $\alpha_g \alpha_h(x) = \alpha_{gh}(x)$ for all $x \in D_{(gh)^{-1}} \cap D_{h^{-1}}$.

If $D_g = A$ for every $g \in G$, then we say that α is a (global) *action* on G.

Note that condition (3) of Definition 2.1 implies that $\alpha_g^{-1} = \alpha_{g^{-1}}$. Moreover, it is easy to see that condition (2) is equivalent to the equality $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$, for all $g, h \in G$ (see [27, Proposition 2.6]).

Remark 2.2. It is easy to see that defining the composition $\alpha_g \circ \alpha_h$ on $D_{(gh)^{-1}} \cap D_{h^{-1}}$, which is the largest possible domain on which it makes sense, items (2) and (3) in Definition 2.1 say that $\alpha_g \circ \alpha_h$ is a restriction of α_{gh}.

Let $A, B \in \mathcal{C}$. Suppose that A is an ideal of B and that η is a (global) action of G on the algebra B, $\eta_g : B \to B, g \in G$. Then, setting $D_g = A \cap \eta_g(A)$ and $\alpha_g = \eta_g|_{D_{g^{-1}}}$, it is readily seen that $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial action of G on A. This partial action is called *restriction of* η *to* A.

Definition 2.3. Let $A, \widetilde{A} \in \mathcal{C}$ and G be a group. Suppose that $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $\widetilde{\alpha} = (\{\widetilde{D}_g\}_{g\in G}, \{\widetilde{\alpha}_g\}_{g\in G})$ are partial actions of G on A and $\widetilde{A}$, respectively. We say that α is *isomorphic* to $\widetilde{\alpha}$ if there exists an isomorphism $\varphi : A \to \widetilde{A}$ (in $\mathcal{C}$) such that for every $g \in G$ the equality $\varphi \circ \alpha_g = \widetilde{\alpha}_g \circ \varphi$ holds on $D_{g^{-1}}$.

Definition 2.4. Let $A \in \mathcal{C}$ and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on A. We say that the (global) action η of G on B, with $B \in \mathcal{C}$, is a *globalization* of α if there exists a monomorphism $\iota : A \to B$ such that:

(1) $\iota(A)$ is an ideal of B,
(2) $B = \sum_{g\in G} \eta_g(\iota(A))$,
(3) $\iota(D_g) = \iota(A) \cap \eta_g(\iota(A))$,
(4) $\iota \circ \alpha_g = \eta_g \circ \iota$ on $D_{g^{-1}}$.

A globalization for α will be denoted by the pair (η, B).

If (η, B) is a globalization of the partial action α of G on A, then it follows that the map $\iota : A \to \iota(A)$ is an isomorphism between α and the restriction of η to $\iota(A)$. Therefore, whenever (η, B) is a globalization for a partial action α of G on A we may assume, without loss of generality, that A is an ideal of B and that the embedding $\iota : A \to B$ of Definition 2.4 is given by the inclusion.

Following [17] we say that a Lie algebra L is *faithful* if $Z(L) = 0$. Furthermore, an ideal $I \trianglelefteq L$ is said to be a direct factor of L if there exists an ideal J such that $L = I \oplus J$. This is equivalent to the existence of a homomorphism $\pi : L \to L$ of L onto I such that $\pi^2 = \pi$.

In [17, Theorem 3.6] sufficient conditions for the existence and uniqueness of a globalization for a partial group action on some classes of non-associative algebras were given. Analyzing the uniqueness part of the proof of [17, Theorem 3.6], one can easily see that the following theorem holds.

Theorem 2.5. *Let L be a Lie algebra and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a globalizable partial action of G on L. If each D_g is a faithful direct factor of L, then any two faithful globalization of α are isomorphic.*

In analogy with the associative case (see [30, Definition 1.1]) we give the next:

Definition 2.6. Let $A \in \mathcal{C}$ and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on A. We say that α is of *finite type* if there exist $g_1, \dots, g_n \in G$ such that $A = D_{gg_1} + \cdots + D_{gg_n}$ for any $g \in G$.

Proposition 2.7 *([17, Proposition 3.10]). Let $A \in \mathcal{C}$ and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on A with globalization (η, B). Then the following holds.*

(1) If α is of finite type then there exist $g_1, \dots, g_n \in G$ such that $B = \sum_{i=1}^n \eta_{g_i}(A)$.
(2) If A is idempotent and there exist $g_1, \dots, g_n \in G$ such that $B = \sum_{i=1}^n \eta_{g_i}(A)$, then α is of finite type.

3. Semiprime and non-degenerate Lie algebras

An element x of a Lie algebra L is called *absolute zero divisor* if $\operatorname{ad}^2 x = 0$; and L is called *non-degenerate* (or strongly non-degenerate according to Kostrikin) if L has no nonzero absolute zero divisors. For instance, every finite-dimensional semisimple Lie algebra L is non-degenerate since every absolute zero divisor of L is contained in the nilradical of L (see [18, Proposition 2.2.3]), and hence it lies in $\operatorname{rad}(L) = 0$ (the maximal solvable ideal of L). The Kostrikin radical $\mathcal{K}(L)$ of L is defined as the smallest ideal of L such that $L/\mathcal{K}(L)$ is non-degenerate. It is shown, by E. Zelmanov [52, Corollary 1], that $\mathcal{K}(L)$ is hereditary by ideals, that is,

$$\mathcal{K}(I) = \mathcal{K}(L) \cap I \quad \text{for any ideal } I \text{ of } L. \tag{1}$$

In what follows, given an element x in a Lie algebra L we denote by $\langle x^L \rangle$ the smallest ideal of L containing x (see [51, p. 61]), i.e.

$$\langle x^L \rangle = x\mathbb{F} + [x, L] + [[x, L], L] + \cdots .$$

Note that if H is a subalgebra of L, then $\langle x^H \rangle \subseteq \langle x^L \rangle$.

A Lie algebra L is called *prime* if given $I_1, I_2 \trianglelefteq L$, the equality $[I_1, I_2] = 0$ implies $I_1 = 0$ or $I_2 = 0$. An ideal I of L is called *prime* if L/I is prime. A subset M of L is an

m-system of L if for all $x, y \in M$ we have $[\langle x^L \rangle, \langle y^L \rangle] \cap M \neq \emptyset$. Equivalently, $M \subseteq L$ is an m-system if and only if $L \backslash M$ is a prime ideal. The *radical* $\sqrt{I}$ of $I \trianglelefteq L$ is defined as the intersection of all prime ideals of L containing I. The *prime radical* of L is defined by $\sqrt{0}$ and it is denoted by $\mathcal{R}(L)$.

Lemma 3.1 *(see [9, Satz 9]). An element $x \in L$ is in $\sqrt{I}$ if and only if, every m-system M containing x meets I.*

Following [47] we say that a Lie algebra L is *semiprime* if it has no nonzero abelian ideals. This is equivalent to $Z(I) = I \cap C_L(I) = 0$ for every nonzero ideal I of L, where $C_L(I) = \{x \in L \mid [x, I] = 0\}$ is the centralizer of I on L. Thus, a semiprime Lie algebra is a faithful Lie algebra in which every ideal is faithful. For instance, it is easy to see that every non-degenerate Lie algebra is semiprime. An ideal I of a Lie algebra L is called semiprime if the Lie algebra L/I is semiprime.

Lemma 3.2. *Let I be an ideal of a Lie algebra L. The following are equivalent:*

(1) I is semiprime ideal of L;
(2) given $J \trianglelefteq L$, $[J, J] \subseteq I$ implies $J \subseteq I$;
(3) given $x \in L$, $[\langle x^L \rangle, \langle x^L \rangle] \subseteq I$ implies $\langle x^L \rangle \subseteq I$.

Proof. The implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are immediate. For the implication (3) $\Longrightarrow$ (1) let J/I be an abelian ideal of L/I. Then, for $x \in J$ we have that $[\langle x^L \rangle, \langle x^L \rangle] \subseteq I$, and thus $\langle x^L \rangle \subseteq I$. Hence $x \in I$, and $J/I = 0$, as required. □

The theory of prime and semiprime Lie algebras is very similar to that of prime and semiprime associative rings. In particular, analogously to the case of associative rings, it is possible to show that if H is a subalgebra of a Lie algebra L, then $H \cap \mathcal{R}(L) \subseteq \mathcal{R}(H)$. We could not find a reference for this result, and in order to make our exposition self-contained we give a proof.

We say that a subset N of L is an *n-system* of L if for all $x \in N$ we have $[\langle x^L \rangle, \langle x^L \rangle] \cap N \neq \emptyset$. By part (3) of Lemma 3.2, $N \subseteq L$ is an n-system if and only if $L \backslash N$ is a semiprime ideal of L.

Lemma 3.3. *Suppose that an element x of L belongs to an n-system N. Then there exists an m-system $M \subseteq N$ such that x belongs to M.*

Proof. By definition of an n-system there exists a sequence $x_1, x_2, \ldots$, such that $x_1 = x$ and $x_{i+1} \in [\langle x_i^L \rangle, \langle x_i^L \rangle] \cap N$. In particular we have that $\langle x_{i+1}^L \rangle \subseteq \langle x_i^L \rangle$. We claim that $M = \{x_1, x_2, \ldots\}$ is an m-system. Indeed, given $x_i, x_j \in M$ we may suppose that $i < j$. Then $\langle x_j^L \rangle \subseteq \langle x_i^L \rangle$, and thus

$$x_{j+1} \in [\langle x_j^L \rangle, \langle x_j^L \rangle] \subseteq [\langle x_j^L \rangle, \langle x_i^L \rangle];$$

so that M is an m-system. □

Proposition 3.4. *Let I be an ideal of a Lie algebra L. Then I is a semiprime ideal of L if and only if $I = \sqrt{I}$. Consequently, $\sqrt{\sqrt{I}} = \sqrt{I}$.*

Proof. Suppose that I is a semiprime ideal of L. It is enough to verify that $\sqrt{I} \subseteq I$. Let $x \notin I$. Then $N = L \backslash I$ is an n-system with $x \in N$. By Lemma 3.3 there is an m-system $M \subseteq N$ with $x \in M$ and, consequently, $M \cap I = \emptyset$. It follows from Lemma 3.1 that $x \notin \sqrt{I}$. The converse is clear since the intersection of prime (semiprime) ideals is a semiprime ideal. □

Corollary 3.5. *L is semiprime if and only if $\mathcal{R}(L) = 0$.*

Using Corollary 3.5 we can characterize the prime radical $\mathcal{R}(L)$ of L as the smallest ideal of L such that $L/\mathcal{R}(L)$ is a semiprime Lie algebra. Indeed, suppose L/I is a semiprime algebra. Then I is a semiprime ideal of L, and thus $I = \sqrt{I} \supseteq \sqrt{0} = \mathcal{R}(L)$, as claimed. This characterization also allows us to conclude that $\mathcal{R}(L) \subseteq \mathcal{K}(L)$.

Proposition 3.6. *Suppose that H is a subalgebra of L. Then $H \cap \mathcal{R}(L) \subseteq \mathcal{R}(H)$. In particular, if $H \subseteq \mathcal{R}(L)$, then $H = \mathcal{R}(H)$.*

Proof. Let $x \in H \cap \mathcal{R}(L)$. Consider any m-system M of H containing x. By definition, M is also an m-system of L, and by Lemma 3.1 we have $0 \in M$, as $x \in \mathcal{R}(L)$. Therefore $x \in \mathcal{R}(H)$. □

A nonzero ideal I of a Lie algebra L is called *essential* if $I \cap J \neq 0$ for every nonzero ideal J of L. If L is semiprime then it is easy to verify that I is essential if and only if $C_L(I) = 0$. Moreover, $I \oplus C_L(I)$ is always essential. We denote by $\mathcal{E}(L)$ the set of all essential ideals of L. Notice that $\mathcal{E}(L)$ is closed under intersections and products; i.e. if I and J belong to $\mathcal{E}(L)$, then $I \cap J$ and $[I, J]$ also belong to $\mathcal{E}(L)$. The following result deals with some elementary observations about essential ideals.

Lemma 3.7. *Let L be a semiprime Lie algebra and I an ideal of L which is a semiprime Lie algebra. Then, the following holds:*

(1) if $J \in \mathcal{E}(L)$, then $I \cap J \in \mathcal{E}(I)$; and
(2) if $K \in \mathcal{E}(I)$, then $K \oplus C_L(I) \in \mathcal{E}(L)$.

Proof. (1) Let $x, y \in C_I(I \cap J)$. Then, the Jacobi identity

$$[J, [x, y]] + [x, [y, J]] + [y, [J, x]] = 0$$

implies that $[x, y] \in C_L(J) = 0$, since $[y, J], [J, x] \subseteq I \cap J$. Thus, by the semiprimeness of I, we must have $C_I(I \cap J) = 0$.

(2) Let $0 \neq N$ be an ideal of L. If $0 \neq N \cap I$, then

$$0 \neq N \cap I \cap K = N \cap K \subseteq N \cap (K \oplus C_L(I)).$$

On the other hand, if $N \cap I = 0$ then we have that $N \subseteq C_L(I)$, and therefore $0 \neq N \subseteq N \cap (K \oplus C_L(I))$. $\square$

Let L be a semiprime Lie algebra and I an ideal of L, following [30] we define the *closure* of I as the set

$$[I] = \{x \in L \mid [x, J] \subseteq I \text{ for some } J \in \mathcal{E}(L)\};$$

and we say that I is *closed* if $I = [I]$. It is easy to see that the map $I \mapsto [I]$ is monotone, i.e. $I \subseteq J$ implies $[I] \subseteq [J]$.

Lemma 3.8. *Let L be a semiprime Lie algebra and I, J ideals of L. Then the following holds*

(1) $[I]$ *is an ideal of* L*,*
(2) $[I] = C_L(C_L(I))$*, and*
(3) $[I \cap J] = [I] \cap [J]$*.*

Proof. (1) Let $x \in [I]$ and $y \in L$. Then, for some $J \in \mathcal{E}(L)$ we have $[x, J] \subseteq I$. Thus, the Jacobi identity

$$[J, [x, y]] + [x, [y, J]] + [y, [J, x]] = 0,$$

yields $[J, [x, y]] \subseteq I$, and so $[x, y] \in [I]$.

(2) Let $x \in C_L(C_L(I))$. As $I \oplus C_L(I) \in \mathcal{E}(L)$ and $[x, I \oplus C_L(I)] = [x, I] \subseteq I$, we have that $x \in [I]$. For the opposite inclusion let $x \in [I]$. Then $[x, J] \subseteq I$ for some $J \in \mathcal{E}(L)$. As $[x, [C_L(I), J]] \subseteq [x, C_L(I)] \cap [x, J] \subseteq C_L(I) \cap I = 0$ and $[C_L(I), [J, x]] \subseteq [C_L(I), I] = 0$, the Jacobi identity

$$[J, [x, C_L(I)]] + [x, [C_L(I), J]] + [C_L(I), [J, x]] = 0$$

implies that $[x, C_L(I)] \subseteq C_L(J) = 0$. Hence $x \in C_L(C_L(I))$.

(3) Since the map $I \mapsto [I]$ is monotone, we obtain $[I \cap J] \subseteq [I] \cap [J]$. Conversely, suppose that $x \in [I] \cap [J]$. Then $[x, K_1] \subseteq I$ and $[x, K_2] \subseteq J$ for some $K_1, K_2 \in \mathcal{E}(L)$. Setting $K = K_1 \cap K_2$ we have that $[x, K] \subseteq I \cap J$ for $K \in \mathcal{E}(L)$. Therefore $x \in [I \cap J]$. $\square$

Observe that a closed ideal I of L is a semiprime ideal. Indeed, if J is an ideal of L with $[J, J] \subseteq I$, then $[J \oplus C_L(J), J] \subseteq I$, yielding $J \subseteq [I] = I$.

The following elementary fact will be useful for us.

Lemma 3.9. *Let L be a semiprime Lie algebra and I a nonzero ideal of L. If $\phi, \psi : L \to I$ are two derivations which coincide on I, then $\phi = \psi$.*

Proof. Let $x \in L$ and $y \in I$. Then the equalities

$$[\phi(x), y] + [x, \phi(y)] = \phi([x, y]) = \psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$$

imply that $[\phi(x), y] = [\psi(x), y]$, which amounts to $[\phi(x) - \psi(x), y] = 0$. As $y \in I$ is arbitrary, we obtain $\phi(x) - \psi(x) \in I \cap C_L(I) = 0$. □

4. Maximal algebra of quotients

For convenience of notation, in what follows the second term $[L, L]$ of the derived series of a Lie algebra L will be written also as L^2.

Let L be a semiprime Lie algebra and I an ideal of L. A *partial derivation* of I in L is a linear map $\delta : I \to L$ such that $\delta[x, y] = [\delta x, y] + [x, \delta y]$ for all $x, y \in I$. Note that if $\delta : I \to L$ is a partial derivation of I in L and $J \trianglelefteq L$ is such that $J^2 \subseteq I$, then

$$\delta(J^2) = [\delta(J), J] + [J, \delta(J)] \subseteq J. \tag{2}$$

Let us denote by $\mathrm{PDer}(I, L)$ the set of all partial derivations of I in L. On the set $\mathcal{D} = \{(\delta, I) \mid \delta \in \mathrm{PDer}(I, L), I \in \mathcal{E}(L)\}$ we define the following equivalence relation: $(\delta, I) \equiv (\lambda, J)$ if and only if there is $K \in \mathcal{E}(L)$ with $K \subseteq I \cap J$ and $\delta|_K = \lambda|_K$. Let us denote by δ_I the equivalence class of $(\delta, I) \in \mathcal{D}$. It is shown, in [47], that the quotient set $Q(L) = \mathcal{D}/\equiv$ endowed with the natural vector space structure, and the bracket given by $[\delta_I, \lambda_J] = [\delta, \lambda]_{(I \cap J)^2}$, is a semiprime Lie algebra. Moreover, if L is non-degenerate so is $Q(L)$ (see [47, Proposition 2.7(3)]). The Lie algebra $Q(L)$ is called the *maximal algebra of quotients of L*.

Theorem 4.1 *([47, Theorem 3.8]). Let L be a semiprime Lie algebra which is a subalgebra of S. Then, S is isomorphic to $Q(L)$, under an isomorphism which is the identity on L, if and only if the following conditions are satisfied:*

(1) For any $s \in S$ there exists $I \in \mathcal{E}(L)$ such that $[s, I] \subseteq L$.
(2) For $s \in S$ and $I \in \mathcal{E}(L)$, the equality $[s, I] = 0$ implies $s = 0$.
(3) For any $I \in \mathcal{E}(L)$ and $\delta \in \mathrm{PDer}(I, L)$, there exists $s \in S$ such that $\delta x = [s, x]$ for all $x \in I$.

Let L be a semiprime Lie algebra and I and ideal of L. Following [29] we define the set

$$Q_I = \{q \in Q(L) \mid [q, J] \subseteq I \text{ for some } J \in \mathcal{E}(L)\}.$$

It is easy to see that the map $I \mapsto Q_I$ is monotone, i.e. $I \subseteq J$ implies $Q_I \subseteq Q_J$. It can be seen that Q_I is a subalgebra of $Q(L)$ containing $[I]$. Indeed, if $q_1, q_2 \in Q(L)$ are such that $[q_i, J_i] \subseteq I$ for some $J_i \in \mathcal{E}(L)$, for $i = 1, 2$; then, using (2) and the Jacobi identity one can easily verify that $[[q_1, q_2], (J_1 \cap J_2)^2] \subseteq I$. Also, it follows immediately from the definition that $Q_0 = 0$, $Q_L = Q(L)$, and that

$$Q_I \cap Q_J = Q_{I \cap J}, \quad \text{and} \quad Q_I + Q_J \subseteq Q_{I+J} \tag{3}$$

for any ideals I, J of L.

Lemma 4.2. *Suppose that L is a semiprime Lie algebra and that I is an ideal of L which is a semiprime Lie algebra. Then the following holds:*

(1) If $J \in \mathcal{E}(I)$, then $C_{Q_I}(J) = 0$.
(2) If $J \in \mathcal{E}(I)$ and $\delta : J \to I \in \mathrm{PDer}(J, I)$, then there is $q \in Q_I$ such that $\delta x = [q, x]$ for all $x \in J$ and $[q, C_L(I)] = 0$.

Consequently, $Q_I \cong Q(I)$.

Proof. (1) Let $q \in Q_I$ such that $[q, J] = 0$. We know that there exists $K \in \mathcal{E}(L)$ such that $[q, K] \subseteq I$. Using Lemma 3.7 we have that $K \cap J$ belongs to $\mathcal{E}(I)$ and that $(K \cap J) \oplus C_L(I)$ and $K \cap (J \oplus C_L(I))$ belong to $\mathcal{E}(L)$. Furthermore, $N = (K \cap J) \oplus C_K(I) \in \mathcal{E}(L)$, because

$$(K \cap J) \oplus C_K(I) = ((K \cap J) \oplus C_L(I)) \cap (K \cap (J \oplus C_L(I))).$$

Note that $[q, N] \subseteq I$ and $[N, J] = [K \cap J, J] \oplus [C_K(I), J] = [K \cap J, J] \subseteq J$. Then, the Jacobi identity

$$[J, [q, N]] + [q, [N, J]] + [N, [J, q]] = 0$$

implies that $[q, N] \in C_I(J) = 0$. Since $q \in Q(L)$, Theorem 4.1(2) yields $q = 0$, as required.

(2) By Lemma 3.7(2) $J \oplus C_L(I)$ belongs to $\mathcal{E}(L)$. Let δ^L be the element in $\mathrm{PDer}(J \oplus C_L(I), L)$ defined by $\delta^L(x + y) = \delta(x) \in I$ for all $x \in J$ and $y \in C_L(I)$. In particular, we have $\delta^L|_J = \delta$. Thus, by Theorem 4.1(3), there exists $q \in Q(L)$ such that $[q, z] = \delta^L z$ for all $z \in J \oplus C_L(I)$. Hence, we obtain $[q, x] = \delta x$ for all $x \in J$, and $[q, C_L(I)] = 0$. □

Proposition 4.3. *Suppose that L is a semiprime Lie algebra and that I is an ideal of L which is a semiprime Lie algebra. Then, the following holds:*

(1) $[Q_I, C_L(I)] = 0$.
(2) Q_I is an ideal of $Q(L)$.
(3) If $L = I \oplus C_L(I)$, then $Q(L) = Q_I \oplus Q_{C_L(I)}$. In particular, $Q_{C_L(I)} = C_{Q(L)}(Q_I)$.

Proof. (1) Let $q \in Q_I$. Then, there exists $J \in \mathcal{E}(L)$ such that $[q, J] \subseteq I$. Thus $\mathrm{ad}\,|_{(I\cap J)} \in \mathrm{PDer}(I \cap J, I)$. As, by Lemma 3.7(1), $I \cap J \in \mathcal{E}(I)$, Lemma 4.2(2) gives $\widetilde{q} \in Q_I$, so that $[q, x] = [\widetilde{q}, x]$ for all $x \in I \cap J$ and $[\widetilde{q}, C_L(I)] = 0$. Hence $\widetilde{q} - q \in C_{Q_I}(I \cap J) = 0$ (see Lemma 4.2(1)), yielding $q = \widetilde{q}$. Therefore $[q, C_L(I)] = 0$.

(2) Let $q \in Q_I$ and $p \in Q(L)$. Then there exist $J, K \in \mathcal{E}(L)$ such that $[q, J] \subseteq I$ and $[p, K] \subseteq L$. Using Lemma 3.7 we have that $(I \cap J) \oplus C_L(I)$ and $(K \cap I \cap J) \oplus C_L(I)$ are in $\mathcal{E}(L)$. Thus, the ideal $N = (K \cap I \cap J) \oplus C_K(I)$ belongs to $\mathcal{E}(L)$, as

$$(K \cap I \cap J) \oplus C_K(I) = (K \cap I \cap J) \oplus C_L(I)) \cap (K \cap ((I \cap J) \oplus C_L(I))).$$

Considering that $[q, C_L(I)] = 0$ (by item (1)), and using equality (2), we see that

$$[p, [q, N^4]] = [p, [q, (K \cap I \cap J)^4]] \subseteq [p, (K \cap I \cap J)^2] \subseteq K \cap I \cap J$$

and

$$[q, [p, N^4]] \subseteq [q, N^2] = [q, (K \cap I \cap J)^2] \subseteq K \cap I \cap J.$$

So $[[q, p], N^4] = [p, [q, N^4]] - [q, [p, N^4]] \subseteq I$; showing that $[q, p] \in Q_I$.

(3) Let $q \in Q(L)$. Then, there is $K \in \mathcal{E}(L)$ such that $[q, K] \subseteq L$. By equality (2), $\mathrm{ad}\, q((I \cap K)^2) \subseteq I \cap K$, and then the mapping $\mathrm{ad}\, q|_{(I\cap K)^2}$ belongs to $\mathrm{PDer}(I \cap K, I)$. As $(I \cap K)^2 \in \mathcal{E}(I)$ (see Lemma 3.7(1)), Lemma 4.2(2) provides $p \in Q_I$ such that $[p, x] = [q, x]$ for all $x \in (I \cap K)^2$ and $[p, C_L(I)] = 0$. In an analogous way, there exists $r \in Q_{C_L(I)}$ so that $[r, y] = [q, y]$ for all $y \in C_K(I)^2 = (K \cap C_L(I))^2 \in \mathcal{E}(L)$ and $[r, I] = 0$. Now, note that $(I \cap K)^2 \oplus C_K(I)^2$ belongs to $\mathcal{E}(L)$, since

$$(I \cap K)^2 \oplus C_K(I)^2 = ((I \cap K)^2 \oplus C_L(I)) \cap (C_K(I)^2 \oplus I).$$

Furthermore, for all $x \in (I \cap K)^2$ and $y \in C_K(I)^2$, we have

$$[p + r, x + y] = [p, x] + [r, y] = [q, x] + [q + y] = [q, x + y].$$

Theorem 4.1(2) implies the equality $q = p + r$. So $Q(L) = Q_I + Q_{C_L(I)}$. Now, note that $Q_I \cap Q_{C_L(I)} = Q_{I\cap C_L(I)} = Q_0 = 0$. This yields $Q(L) = Q_I \oplus Q_{C_L(I)}$. Finally, note that the semiprimeness of $Q(L)$ (see [47, Proposition 3.6]) implies the equality $Q_{C_L(I)} = C_{Q(L)}(Q_I)$. □

Suppose that I, J are two ideals of L which are semiprime Lie algebras, and that $\phi : I \to J$ is an isomorphism. Let $q \in Q_I$. Then, there exists $K \in \mathcal{E}(L)$ such that $[q, K] \subseteq I$. Furthermore, $\mathrm{ad}\, q|_{I\cap K}$ belongs to $\mathrm{PDer}(I \cap K, I)$, and therefore $\phi \circ \mathrm{ad}\, q|_{I\cap K} \circ \phi^{-1}$ belongs to $\mathrm{PDer}(\phi(I \cap K), J)$. By Lemma 3.7(1) $I \cap K \in \mathcal{E}(I)$, so $\phi(I \cap K) \in \mathcal{E}(J)$. In virtue of Lemma 4.2(2) there is $\phi^*(q) \in Q_J$ such that

$$\operatorname{ad}\phi^*(q) = \phi \circ \operatorname{ad} q|_{I\cap K} \circ \phi^{-1} \quad \text{ on } \quad \phi(I \cap K).$$

It is easy to see that $\phi^*(q)$ does not depend on the ideal $K \in \mathcal{E}(L)$. Moreover, $\phi^*(q)$ satisfy the following property:

(P) There is $N \in \mathcal{E}(I)$ such that $[q, N] \subseteq I$ and $\operatorname{ad}\phi^*(q) = \phi \circ \operatorname{ad} q|_N \circ \phi^{-1}$ on $\phi(N)$.

Property (P) determines $\phi^*(q)$, in the sense that if $\widetilde{q}$ is an element in Q_J satisfying property (P), then $\widetilde{q} = \phi^*(q)$. Indeed, suppose that there is $N \in \mathcal{E}(I)$ such that $\operatorname{ad}\widetilde{q} = \phi \circ \operatorname{ad} q|_N \circ \phi^{-1}$ on $\phi(N)$. Then $\operatorname{ad}\phi^*(q) = \operatorname{ad}\widetilde{q}$ on $\phi(I \cap K) \cap \phi(N) \in \mathcal{E}(J)$. Thus Lemma 4.2(1) provides $\phi^*(q) = \widetilde{q}$.

Lemma 4.4. *If $\phi : I \to J$ is an isomorphism, then so is $\phi^* : Q_I \to Q_J$.*

Proof. First we show that ϕ^* is a homomorphism. For let $q_1, q_2 \in Q_I$. Then there exists $N_i \in \mathcal{E}(I)$ such that $[q_i, N_i] \subseteq I$ and $\operatorname{ad}\phi^*(q_i) = \phi \circ \operatorname{ad} q_i|_{N_i} \circ \phi^{-1}$ on $\phi(N_i)$, for $i = 1, 2$. Let $M = N_1 \cap N_2 \in \mathcal{E}(I)$. Then, since $\operatorname{ad} q_i|_M \in \operatorname{PDer}(M, I)$, equality (2) implies that $[q_i, M^2] \subseteq M$, yielding $[q_j, [q_i, M^2]] \subseteq I$ for $i, j \in \{1, 2\}$. So, the Jacobi identity implies that $[[q_1, q_2], M^2] \subseteq I$. On the other hand, it is easy to notice that on the ideal $\phi(M^2) \in \mathcal{E}(J)$ the following equalities hold

$$\begin{aligned}\operatorname{ad}([\phi^*(q_1), \phi^*(q_2)]) &= [\operatorname{ad}\phi^*(q_1), \operatorname{ad}\phi^*(q_2)] \\ &= [\phi \circ \operatorname{ad} q_1|_{K_1} \circ \phi^{-1}, \phi \circ \operatorname{ad} q_2|_{K_2} \circ \phi^{-1}] \\ &= \phi \circ \operatorname{ad}([q_1, q_2])|_{M^2} \circ \phi^{-1}.\end{aligned}$$

Then, $[\phi^*(q_1), \phi^*(q_2)] \in Q_J$ satisfies the property (P), and therefore we have that $[\phi^*(q_1), \phi^*(q_2)] = \phi^*([q_1, q_2])$.

We shall prove now that ϕ^* is injective and onto. Let $q \in Q_I$ such that $\phi^*(q) = 0$. Then, there exists $N \in \mathcal{E}(I)$ such that $[q, N] \subseteq I$ and $0 = \phi \circ \operatorname{ad} q|_N \circ \phi^{-1}$ on $\phi(N)$, which yields $\operatorname{ad} q|_N = 0$. Lemma 4.2(1) implies that $q = 0$; showing that ϕ^* is injective. On the other hand, let $\widetilde{q} \in Q_J$. Then, we can choose $K \in \mathcal{E}(J)$ such that $[\widetilde{q}, K] \subseteq J$. Since $\operatorname{ad}\widetilde{q}|_K \in \operatorname{PDer}(K, J)$, it follows that $\phi^{-1} \circ \operatorname{ad}\widetilde{q}|_K \circ \phi$ belongs to $\operatorname{PDer}(\phi^{-1}(K), I)$. Thus, by Lemma 4.2(2), there is $q \in Q_I$ such that $\operatorname{ad} q|_{\phi^{-1}(K)} = \phi^{-1} \circ \operatorname{ad}\widetilde{q}|_K \circ \phi$, which yields $\phi^*(\widetilde{q}) = q$. Hence ϕ^* is onto. □

5. Partial group actions on semiprime Lie algebras

This section is devoted to partial group actions on a semiprime Lie algebra and its maximal algebra of quotients. The main result here is inspired by [15, Theorem 2.3], where one can find sufficient conditions for the existence and uniqueness (up to isomorphism) of a semiprime globalization for a partial group action on a semiprime ring.

However, there is a gap in the uniqueness part of the proof of that result. So, the purpose of this section is to give a version of the above-mentioned fact for the existence of a semiprime (non-degenerate) globalization of a partial group action on a semiprime (non-degenerate) Lie algebra, and show that with an additional reasonable condition the uniqueness of such a globalization holds.

Proposition 5.1. *Let L be a semiprime Lie algebra and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a partial action of G on L where each D_g is a semiprime Lie algebra. Set $Q_g = Q_{D_g}$ and let $\alpha^*_g : Q_{g^{-1}} \to Q_g$ be the isomorphism induced by α_g. Then $\alpha^* = (\{Q_g\}_{g\in G}, \{\alpha^*_g\}_{g\in G})$ is a partial action of G on $Q(L)$.*

Proof. It is obvious that $Q_e = Q(L)$ and that α^*_e is the identity map on $Q(L)$ and that

$$\alpha^*_g(Q_{g^{-1}} \cap Q_h) = \alpha^*_g(Q_{D_{g^{-1}}\cap D_h}) = Q_{\alpha_g(D_{g^{-1}}\cap D_h)} = Q_{D_g\cap D_{gh}} = Q_g \cap Q_{gh}.$$

It remains to verify that $\alpha^*_g\alpha^*_h = \alpha^*_{gh}$ on $Q_{(gh)^{-1}} \cap Q_{h^{-1}}$. In order to see this let $q \in Q_{(gh)^{-1}} \cap Q_{h^{-1}}$. By property (P), there is $N \in \mathcal{E}(D_{(gh)^{-1}} \cap D_{h^{-1}})$ such that $[q, N] \subseteq D_{(gh)^{-1}} \cap D_{h^{-1}}$ and $\operatorname{ad}\alpha^*_h(q) = \alpha_h \circ \operatorname{ad} q|_N \circ \alpha_{h^{-1}}$ on $\alpha_h(N)$. Moreover, there exists $K \in \mathcal{E}(D_{g^{-1}} \cap D_h)$ so that $[\alpha^*_h(q), K] \subseteq D_{g^{-1}} \cap D_h$ and $\operatorname{ad}\alpha^*_g(\alpha^*_h(q)) = \alpha_g \circ \operatorname{ad}\alpha^*_h(q)|_K \circ \alpha_{g^{-1}}$ on $\alpha_g(K)$. Let $M = \alpha_{h^{-1}}(K) \cap N \in \mathcal{E}(D_{(gh)^{-1}} \cap D_{h^{-1}})$. Then $\alpha_h(M) \in \mathcal{E}(D_{g^{-1}} \cap D_h)$ and $\alpha_{gh}(M) = \alpha_g \circ \alpha_h(M) \in \mathcal{E}(D_g \cap D_{gh})$. Clearly, $[q, M] \subseteq D_{(gh)^{-1}} \cap D_{h^{-1}}$. Also, it is easy to notice that on $\alpha_{gh}(M)$ we have the equalities

$$\alpha^*_g\alpha^*_h(q) = \alpha_g \circ \operatorname{ad}\alpha^*_h(q)|_K \circ \alpha_{g^{-1}} = \alpha_g \circ \alpha_h \circ \operatorname{ad} q|_M \circ \alpha_{h^{-1}} \circ \alpha_{g^{-1}} = \alpha_{gh} \circ \operatorname{ad} q|_M \circ \alpha_{(gh)^{-1}}.$$

That is, $\alpha^*_g\alpha^*_h(q) \in Q_{D_g\cap D_{gh}}$ satisfies property (P), and thus $\alpha^*_g\alpha^*_h(q) = \alpha^*_{gh}(q)$. □

Lemma 5.2. *Let L be an arbitrary Lie algebra and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a partial action of G on L. If α possesses a globalization, then for every $a \in L$ and $g \in G$ there exists $\gamma_g(a) \in \operatorname{Der}(L)$ such that:*

(1) $\gamma_g(a)(L) \subseteq D_g$, and
(2) $\gamma_g(a) = \alpha_g \circ \operatorname{ad} a \circ \alpha_{g^{-1}}$ on D_g.

Proof. Suppose that (H, β) is a globalization for α. We will verify that $\gamma_g(a) = \operatorname{ad}\beta_g(a)$ satisfies conditions (1) and (2). Given $x \in D_g$ we have

$$\alpha_g \circ \operatorname{ad} a \circ \alpha_{g^{-1}}(x) = \beta_g([a, \beta_{g^{-1}}(x)]) = [\beta_g(a), x] = \operatorname{ad}\beta_g(a)(x).$$

Moreover, for $y \in L$ we see that $\operatorname{ad}\beta_g(a)(y) = [\beta_g(a), y] \in \beta_g(L) \cap L = D_g$. □

Proposition 5.3. *Let L be a semiprime Lie algebra and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a partial action of G on L. Suppose that every ideal D_g is closed and each derivation $\gamma_g(a) \in$* $\operatorname{Der}(L)$ *satisfies (1) and (2) of Lemma 5.2. Then*

(1) $\gamma_g(a)(D_h) \subseteq D_g \cap D_h$ *for any* $a \in L$*,*
(2) $\gamma_g(a)(L) \subseteq D_g \cap D_{gh}$ *for any* $a \in D_h$*,*
(3) $[\gamma_g(a), \gamma_g(b)] = \gamma_g([a,b])$ *for any* $a, b \in L$*,*
(4) $\gamma_h(\gamma_g(b)a) = [\gamma_h(a), \gamma_{hg}(b)]$*, for any* $a, b \in L$*, and*
(5) $\gamma_h(a) = \gamma_{hg^{-1}}(\alpha_g(a))$ *for any* $a \in D_{g^{-1}}$*.*

Proof. (1) Let $x \in D_h$. For every $y \in D_g$ we have that $\gamma_g(a)([x,y]) = \alpha_g([a, \alpha_{g^{-1}}([x,y])])$ and $[x, \gamma_g(a)y]$ belongs to $D_g \cap D_h$. This implies that

$$[\gamma_g(a)x, y] = \gamma_g(a)([x,y]) - [x, \gamma_g(a)y] \in D_g \cap D_h.$$

Now, for every $z \in C_L(D_g)$ we have that $[\gamma_g(a)x, z] = 0 \in D_g \cap D_h$. It follows that $[\gamma_g(a)x, D_g \oplus C_L(D_g)] \subseteq D_g \cap D_h$ and, consequently, $\gamma_g(a)x \in [D_g \cap D_h] = D_g \cap D_h$.

(2) Let $x \in L$. For any $y \in D_g$ we see that

$$\gamma_g(a)y = \alpha_g([a, \alpha_{g^{-1}}(y)]) \in \alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh},$$

and therefore $[\gamma_g(a)x, y] = \gamma_g(a)([x,y]) - [x, \gamma_g(a)y] \in D_g \cap D_{gh}$. As for every $z \in C_L(D_g)$ we have that $[\gamma_g(a)x, z] = 0 \in D_g \cap D_h$, it follows that $[\gamma_g(a)x, D_g \oplus C_L(D_g)] \subseteq D_g \cap D_h$. Hence, $\gamma_g(a)x \in [D_g \cap D_h] = D_g \cap D_h$.

(3) By Lemma 3.9 it is enough to verify that $[\gamma_g(a), \gamma_g(b)] = \gamma_g([a,b])$ on D_g. For note that on D_g holds

$$[\gamma_g(a), \gamma_g(b)] = [\alpha_g \circ \operatorname{ad} a \circ \alpha_{g^{-1}}, \alpha_g \circ \operatorname{ad} b \circ \alpha_{g^{-1}}] = \\ = \alpha_g \circ ([\operatorname{ad} a, \operatorname{ad} b]) \circ \alpha_{g^{-1}} = \alpha_g \circ \operatorname{ad}([a,b]) \circ \alpha_{g^{-1}} = \gamma_g([a,b]).$$

(4) Let us denote by I the ideal $D_h \cap D_{hg}$ of L. By items (1) and (2) we have that $\gamma_h(\gamma_g(b)a)(I) \subseteq I$. On the other hand, it is easy to see that $[\gamma_h(a), \gamma_{hg}(b)](I) \subseteq I$. Thus, by Lemma 3.9, it is enough to check that $\gamma_h(\gamma_g(b)a)$ and $[\gamma_h(a), \gamma_{hg}(b)]$ coincide on I. On I we have that

$$[\gamma_h(a), \gamma_{hg}(b)] = (\alpha_h \circ \operatorname{ad} a \circ \alpha_{h^{-1}}) \circ (\alpha_{hg} \circ \operatorname{ad} b \circ \alpha_{(hg)^{-1}}) - \\ - (\alpha_{hg} \circ \operatorname{ad} b \circ \alpha_{(hg)^{-1}}) \circ (\alpha_h \circ \operatorname{ad} a \circ \alpha_{h^{-1}}).$$

Using that $\alpha_{hg} = \alpha_h \circ \alpha_g$ on $D_{g^{-1}} \cap D_{(hg)^{-1}}$ and $\alpha_{(hg)^{-1}} = \alpha_{g^{-1}} \circ \alpha_{h^{-1}}$ on $D_h \cap D_{hg}$, we see on I:

$$\begin{aligned} [\gamma_h(a), \gamma_{hg}(b)] &= \alpha_h \circ \operatorname{ad} a \circ \alpha_g \circ \operatorname{ad} b \circ \alpha_{g^{-1}} \circ \alpha_{h^{-1}} - \alpha_h \circ \alpha_g \circ \operatorname{ad} b \circ \alpha_{g^{-1}} \circ \operatorname{ad} a \circ \alpha_{h^{-1}} \\ &= \alpha_h \circ \operatorname{ad} a \circ \gamma_g(b) \circ \alpha_{h^{-1}} - \alpha_h \circ \gamma_g(b) \circ \operatorname{ad} a \circ \alpha_{h^{-1}} \\ &= \alpha_h \circ [\operatorname{ad} a \circ \gamma_g(b)] \circ \alpha_{h^{-1}} \\ &= \alpha_h \circ \operatorname{ad}(\gamma_g(b)a) \circ \alpha_{h^{-1}} \end{aligned}$$

$$= \gamma_h(\gamma_g(b)a),$$

as required.

(5) Denote by I the ideal $D_h \cap D_{hg^{-1}}$. Item (2) implies that $\gamma_h(a)(L) \subseteq I$ and $\gamma_{hg^{-1}}(\alpha_g(a))(L) \subseteq I$. Thus, by Lemma 3.9 it is enough to show that $\gamma_h(a)$ coincides with $\gamma_{hg^{-1}}(\alpha_g(a))$ on I. Let $x \in I$. Since $[\alpha_g(a), \alpha_{gh^{-1}}(x)] \in D_g \cap D_{gh^{-1}}$, we see that

$$\gamma_{hg^{-1}}(\alpha_g(a))(x) = \alpha_{hg^{-1}}([\alpha_g(a), \alpha_{gh^{-1}}(x)]) = \alpha_h([a, \alpha_{h^{-1}}(x)]) = \gamma_h(a)(x),$$

as required. □

The proof of Theorem 1.1. The "if" part follows from Lemma 5.2. For the "only if" part let us consider the set $\mathcal{F} = \mathcal{F}(G, \mathrm{Der}(L))$, formed by all functions $f : G \to \mathrm{Der}(L)$. It is easy to see that $\mathcal{F}$ endowed with the bracket

$$[f_1, f_2](g) = [f_1(g), f_2(g)]$$

is a Lie algebra. For convenience of notation, $f(g)$ will be also written as $f|_g$.

Define the global action η of G on $\mathcal{F}$ by

$$\eta_g(f)|_h = f(g^{-1}h),$$

for $f \in \mathcal{F}$ and $h, g \in G$.

Let $\iota : L \to \mathcal{F}$ be the mapping defined by $\iota(a)|_g = \gamma_{g^{-1}}(a)$. We will show that $\iota : L \to \mathcal{F}$ is a well defined monomorphism. By Lemma 3.9, if $\gamma_{g^{-1}}(a)$ and $\widetilde{\gamma}_{g^{-1}}(a)$ are two derivations of L satisfying conditions (1) and (2), then $\gamma_{g^{-1}}(a) = \widetilde{\gamma}_{g^{-1}}(a)$, as they coincide on $D_{g^{-1}}$. Thus, ι is well defined. Proposition 5.3(3) says that ι is a homomorphism. Now, let $a \in L$ such that $\iota(a)|_g = 0$ for every $g \in G$. Then, in particular we have $0 = \gamma_e(a)x = [a, x]$ for all $x \in L$. Thus $a \in Z(L) = 0$, showing that ι is injective.

We will prove that the action η on

$$H = \sum_{g \in G} \eta_g(\iota(L))$$

is a globalization for the partial action α. Firstly, we check that $\iota(L)$ is an ideal of H. For it is enough to verify that $[\iota(L), \eta_g(\iota(L))] \subseteq \iota(L)$ for any $g \in G$. If $a, b \in L$ and $g \in G$, then Lemma 3.9(4) implies that

$$[\iota(a), \eta_g(\iota(a))]|_h = [\iota(a)|_h, \iota(b)|_{g^{-1}h}] = [\gamma_{h^{-1}}(a), \gamma_{h^{-1}g}(b)] = \gamma_{h^{-1}}(\gamma_g(b)a) = \iota(\gamma_g(b)a)|_h$$

for every $h \in G$. Thus, $[\iota(x), \eta_g(\iota(y))] = \iota(\gamma_g(b)a) \in \iota(L)$, and it follows that $[\iota(L), \eta_g(\iota(L))] \subseteq \iota(L)$, as desired.

As a second step, we verify that

$$\iota(L) \cap \eta_g(\iota(L)) = \iota(D_g).$$

Indeed, let $\iota(a) = \eta_g(\iota(b)) \in \iota(L) \cap \eta_g(\iota(L))$ for some $a, b \in L$. Then,

$$\gamma_e(a) = \iota(a)|_e = \eta_g(\iota(b))|_e = \iota(b)|_{g^{-1}} = \gamma_g(b),$$

and this implies $[a, L] = \gamma_e(a)(L) = \gamma_g(b)(L) \subseteq D_g$, yielding $a \in [D_g] = D_g$. Hence, $\iota(L) \cap \eta_g(\iota(L)) \subseteq \iota(D_g)$. Conversely, let $a \in D_g$, and set $b = \alpha_{g^{-1}}(a)$. Then, using Proposition 5.3(5) we obtain that

$$\iota(a)|_h = \gamma_{h^{-1}}(\alpha_g(b)) = \gamma_{h^{-1}g}(b) = \eta_g(\iota(b))|_h,$$

which yields the inclusion $\iota(D_g) \subseteq \iota(L) \cap \eta_g(\iota(L))$. So, it only remains to prove that $\eta_g(\iota(b)) = \iota(\alpha_g(b))$ for $g \in G$ and $b \in D_{g^{-1}}$. This is equivalent to show that $\gamma_{h^{-1}g}(b) = \gamma_{h^{-1}}(\alpha_g(b))$, which once again follows by Proposition 5.3(5). Hence, (η, H) is a globalization for α.

We shall prove now that it is possible to choose a semiprime (non-degenerate) globalization for α. Suppose that (η, H) is any globalization for α. Let us consider the Lie algebra

$$\overline{H} = \begin{cases} H/\mathcal{R}(H), & \text{if } L \text{ is semiprime} \\ H/\mathcal{K}(H), & \text{if } L \text{ is non-degenerate} \end{cases}.$$

Since $\mathcal{K}(H)$ and $\mathcal{R}(L)$ are invariant under automorphisms, η induces a global action on the Lie algebra $\overline{H}$, which we denote by $\overline{\eta}$. One must note that $\overline{\eta}_g \circ \varphi = \varphi \circ \eta_g$ for any $g \in G$. We claim that $(\overline{\eta}, \overline{H})$ is a globalization for α. To see this, first we notice that $\mathcal{R}(H) \cap L \subseteq \mathcal{R}(L) = 0$ if L is semiprime (see Proposition 3.6); and $\mathcal{K}(H) \cap L = \mathcal{K}(L) = 0$ if L is non-degenerate (see (1)). Hence, the restriction of the natural projection $\varphi : H \to \overline{H}$ to L is a monomorphism. In order to show that $(\overline{\eta}, \overline{H})$ is a globalization for α we will verify that the embedding $\varphi\,|_L : L \to \overline{H}$ verifies the conditions (1)-(4) of Definition 2.4. Since items (1), (2), and (4) are immediate, we only need to prove that $\varphi(D_g) = \varphi(L) \cap \overline{\eta}_g(\varphi(L))$. The inclusion $\varphi(D_g) \subseteq \varphi(L) \cap \overline{\eta}_g(\varphi(L))$ is clear. Let us show the converse inclusion when L is non-degenerate. Let $x + \mathcal{K}(H) = \eta_g(y) + \mathcal{K}(H) \in \varphi(L) \cap \overline{\eta}_g(\varphi(L))$, where $x, y \in L$. Then, for every $z \in L$ we have $[x - \eta_g(y), z] = \mathcal{K}(H) \cap L = \mathcal{K}(L) = 0$ (by (1)), and this implies that $[x, z] = [\eta_g(y), z] \in \eta_g(L) \cap L = D_g$. Hence, $[x, L] \subseteq D_g$, yielding $x \in [D_g] = D_g$, as required. The proof of the inclusion in the case when L is semiprime follows from the previous one changing $\mathcal{K}(L)$ to $\mathcal{R}(L)$. The theorem is proved. □

Corollary 5.4. *Suppose that L is a semiprime (respectively, non-degenerate) Lie algebra and that $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial action of G on L in which every D_g*

is a direct factor of L. Then α possesses a semiprime (respectively, non-degenerate) globalization, which is unique up to isomorphism.

Proof. Since L is semiprime (respectively, non-degenerate) we have that $L = D_g \oplus C_L(D_g)$ for each $g \in G$. In particular, D_g is a closed ideal of L. Let $\pi_g : L \to L$ be the Lie homomorphism sending $a \in L$ to the unique element $\pi_g(a)$ in D_g such that $a - \pi_g(a) \in C_L(D_g)$, and consider $\gamma_g(a) = \mathrm{ad}(\alpha_g \pi_{g^{-1}}(a)))$ in $\mathrm{Der}(L)$. Clearly $\gamma_g(a)(L) \subseteq D_g$, and for $x \in D_g$ we see that

$$\alpha_g \circ \mathrm{ad}\, a \circ \alpha_{g^{-1}}(x) = \alpha_g([a, \alpha_{g^{-1}}(x)]) = \alpha_g([\pi_{g^{-1}}(a), \alpha_{g^{-1}}(x)]) = [\alpha_g \pi_{g^{-1}}(a), x] = \gamma_g(a)x.$$

By Theorem 1.1, α possesses a semiprime (or non-degenerate) globalization. The assertion about the uniqueness follows from Theorem 2.5, since a semiprime (or non-degenerate) Lie algebra is faithful. □

Example 5.5. (cf. [46, Theorem 4.3]) Let L be a Lie algebra of the form $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$, where each L_λ is an simple ideal of L. Examples of such Lie algebras are the finite-dimensional Lie algebras with non-degenerate Killing form (see [18, Lemma 4.3.5]), in particular, finite-dimensional semisimple Lie algebras over a field of characteristic 0. One can verify that every ideal of L is a direct factor. More precisely, for an ideal I of L we have that $I = \bigoplus_{\lambda \in \Lambda_1} L_\lambda$, where $\Lambda_1 = \{\lambda \in \Lambda \mid L_\lambda \subseteq I\}$. So, $Z(I) = 0$ for any ideal $I \trianglelefteq L$, and therefore L is semiprime. Applying Corollary 5.4 we conclude that every partial group action α on L has a semiprime globalization that is unique up to isomorphism. But, following the proof of [46, Lemma 2.2(2)], it is easy to see that any globalization of α is semiprime. We conclude then that any partial action α of a group G on L is globalizable, every globalization of α is semiprime and all globalizations of α are isomorphic.

Corollary 5.6. *Suppose that L is a semiprime (respectively, non-degenerate) Lie algebra and that α is as in Corollary 5.4. Then, the partial action $\alpha^* = (\{Q_g\}_{g \in G}, \{\alpha_g^*\}_{g \in G})$ of G on $Q(L)$ induced by α admits a semiprime (respectively, non-degenerate) globalization that is unique up to isomorphism.*

Proof. The semiprimeness of L implies that $L = D_g \oplus C_L(D_g)$. Thus, by Proposition 4.3(3), Q_g is a direct factor of $Q(L)$. So, α^* satisfies the condition of Corollary 5.4. Therefore α^* admits a semiprime (respectively, non-degenerate) globalization that is unique up to isomorphism. □

6. Partial actions on unital special universal envelopes for Jordan algebras

Throughout this section J denotes a unital Jordan algebra and the ground field has characteristic different from 2.

Let A be a unital associative algebra. The underlying vector space of A together with the Jordan product $x \cdot y = (xy + yx)/2$ forms the Jordan algebra A^+. A *unital specialization* of J on A is a Jordan homomorphism $\sigma : J \to A^+$ such that $\sigma(1) = 1$. A pair (A, σ), where A is a unital associative algebra and σ is a unital specialization of J on A, is called a *unital special universal envelope for J* if the following universal property is satisfied: given a unital associative algebra B and a unital specialization $f : J \to B^+$ there exists a unique algebra homomorphism $\widetilde{f} : A \to B$ such that $\widetilde{f} \circ \sigma = f$ and $\widetilde{f}(1) = 1$. It is known that (A, σ) is unique up to isomorphism, and that A is generated as algebra by $\sigma(J)$, the image of J under σ (see [35]).

Theorem 6.1 *(see [35, Theorem 5, p.73]). Let J be a unital Jordan algebra such that $J = J_1 \oplus J_2$, where J_1, J_2 are ideals of J with unities $1_1, 1_2$, respectively. Suppose that (A, σ) is the unital special universal envelope for J. Then, setting $A_i = \sigma(1_i)A$ and $\sigma_i = \sigma|_{A_i}$, we have that (A_i, σ_i) is the unital special universal envelope for A_i, for $i = 1, 2$. Moreover $A = A_1 \oplus A_2$.*

Let $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on J where each D_g is a unital ideal of J with unity 1_g, and let (A, σ) be the unital special universal envelope for J. By [46, Proposition 2.14] each D_g is a direct factor of J. Thus, if $A_g = \sigma(1_g)A$ and σ denotes the restriction of σ to D_g, Theorem 6.1 implies that (A_g, σ) is the unital special universal envelope for D_g. Note that $\sigma(1_g)$ is the unity of A_g, and that A_g is generated by $\sigma(D_g)$. The universal property of (A_g, σ) provides a unique isomorphism $\widetilde{\alpha}_g : A_{g^{-1}} \to A_g$ which verifies $\widetilde{\alpha}_g \circ \sigma = \sigma \circ \alpha_g$, for every $g \in G$.

Proposition 6.2. *With the above notation, we have that $\widetilde{\alpha} = (\{A_g\}_{g\in G}, \{\widetilde{\alpha}_g\}_{g\in G})$ is partial action of G on A. Moreover, $\widetilde{\alpha}$ admits a globalization that is unique up to isomorphism.*

Proof. First, the equalities $A_e = A$ and $\widetilde{\alpha}_e = id_{A_e}$ follow immediately from the universal property. Now, it is easy to see that $A_g \cap A_h$ is unital with unity $\sigma(1_g)\sigma(1_h) = \sigma(1_g 1_h)$, for any $g, h \in G$. Hence $A_g \cap A_h = \sigma(1_g 1_h)A$ is generated by $\sigma(D_g \cap D_h)$. With this in mind we have the following equality

$$\widetilde{\alpha}_g \circ \sigma(D_{g^{-1}} \cap D_h) = \sigma \circ \alpha_g(D_{g^{-1}} \cap D_h) = \sigma(D_{gh} \cap D_g),$$

which implies that $\widetilde{\alpha}_g(A_{g^{-1}} \cap A_h) = A_{gh} \cap A_g$, since $A_{gh} \cap A_g$ is generated by $\sigma(D_{gh} \cap D_g)$. It remains to verify that $\widetilde{\alpha}_h \widetilde{\alpha}_g = \widetilde{\alpha}_{hg}$ on $A_{(hg)^{-1}} \cap A_{g^{-1}}$. To see this note that for $x \in D_{(hg)^{-1}} \cap D_{g^{-1}}$ we have that

$$\widetilde{\alpha}_h \widetilde{\alpha}_g(\sigma(x)) = \widetilde{\alpha}_h(\sigma\alpha_g)(x) = \sigma(\alpha_h \alpha_g(x)) = \sigma(\alpha_{hg}(x)) = \widetilde{\alpha}_{hg}(\sigma(x)).$$

Since $\sigma(D_{(hg)^{-1}} \cap D_{g^{-1}})$ generates $A_{(hg)^{-1}} \cap A_{g^{-1}}$ we conclude that $\widetilde{\alpha}_h \widetilde{\alpha}_g = \widetilde{\alpha}_{hg}$ on $A_{(hg)^{-1}} \cap A_{g^{-1}}$. So, $\widetilde{\alpha}$ is a partial action of G on A.

The last assertion follows from [22, Theorem 4.5], since every A_g is unital. $\square$

Let $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on a unital Jordan algebra J, where each D_g is a unital ideal of J. Then, by [46, Corollary 4.5], α possesses a globalization which is unique up to isomorphism. Suppose that (η, K) is a globalization for $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Then K does not necessarily have a unity, and therefore we cannot say anything about its unital special universal envelope. But, we can improve this assuming that α is of finite type. Indeed, if α is of finite type, Proposition 2.7 provides $g_1, \ldots, g_n \in G$ such that

$$K = \sum_{i=1}^{n} \eta_{g_i}(J),$$

and thus, by [46, Lemma 2.15], K is a unital Jordan algebra.

Theorem 6.3. *Let $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of finite type of G on a unital Jordan algebra J, where each D_g is a unital ideal of J. Suppose that (η, K) is a globalization for α, and that (B, σ) is the unital special universal envelope for K. Then the action $(\widetilde{\eta}, B)$, induced by (η, K), is a globalization for the partial action $\widetilde{\alpha} = (\{A_g\}_{g\in G}, \{\widetilde{\alpha}_g\}_{g\in G})$ of G on the unital special universal envelope for J.*

Proof. Let us denote by 1_g the unity of D_g. By [46, Proposition 2.14] D_g is a direct factor of K. Moreover, since $D_g = J \cap \eta_g(J)$, relations (5) from [46] imply that $1_g = 1_e\eta_g(1_e)$. Thus, in virtue of Theorem 6.1, the unital special universal envelope for D_g is given by (A_g, σ), where σ denotes the restriction of $\sigma : K \to B^+$ to D_g, and $A_g = \sigma(1_g)B$. As we noted above (see the paragraph before Theorem 6.1), A_g is generated, as a subalgebra, by $\sigma(D_g)$. Now, since α is of finite type, Proposition 2.7 provides $g_1, \ldots, g_n \in G$ such that $K = \sum_{i=1}^{n} \eta_{g_i}(J)$. Thus,

$$\sigma(K) = \sum_{i=1}^{n} \sigma\eta_{g_i}(J) = \sum_{i=1}^{n} \widetilde{\eta}_{g_i}\sigma(J) \subseteq \sum_{i=1}^{n} \widetilde{\eta}_{g_i}(A)$$

implies that $B = \sum_{i=1}^{n} \widetilde{\eta}_{g_i}(A)$, since $\sigma(K)$ generates B. As a second step, we show that $A_g = A \cap \widetilde{\eta}_g(A)$. Since σ is a unital specialization, the equality $1_g = 1_e\eta_g(1_e)$ implies that

$$\sigma(1_g) = \sigma(1_e\eta_g(1_e)) = \sigma(1_e)\sigma(\eta_g(1_e)) = \sigma(1_e)\widetilde{\eta}_g(\sigma(1_e)),$$

that is, the unity of A_g coincides with the unity of $A \cap \widetilde{\eta}_g(A)$. Hence, we have that $A_g = \sigma(1_g)B = (\sigma(1_e)\widetilde{\eta}_g(\sigma(1_e))B = A \cap \widetilde{\eta}_g(A)$. Finally we must show that $\widetilde{\alpha}_g = \widetilde{\eta}_g|_{A_{g^{-1}}}$. In order to see this we recall from the universal property of (A_g, σ) that $\widetilde{\alpha}_g$ is the unique isomorphism between $A_{g^{-1}}$ and A_g so that $\widetilde{\alpha}_g \circ \sigma = \sigma \circ \alpha_g$ on $D_{g^{-1}}$. But, as for all $x \in D_{g^{-1}}$ we have

$$\widetilde{\eta}_g\sigma(x) = \sigma\eta_g(x) = \sigma\alpha_g(x),$$

we conclude that $\widetilde{\alpha}_g = \widetilde{\eta}_g|_{A_{g^{-1}}}$. Therefore, $(\widetilde{\eta}, B)$ is a globalization for $\widetilde{\alpha}$. □

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors thank the anonymous referee for the useful suggestions. The first named author was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (Fapesp), process n°: 2020/16594-0, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), process n°: 312683/2021-9. The second named author was supported by FAPESP, process n°: 2019/08659-8.

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